AN ARITHMETIC FUNCTION OF TWO VARIABLES

P.A.Gustomesov

Abstract

The arithmetic function of two variables, $Ns\,(a,n)$, is defined. For all positive integers n and non-negative integers a

$$Ns\,(a,n) = \varphi\,(n)\,\frac{\mu\left(\frac{n}{(a,n)}\right)}{\varphi\left(\frac{n}{(a,n)}\right)},$$

where φ is the Euler function, μ is the Möbius function and (a,n) is the greatest common divisor of integers a and n. Some properties of the function are given along with the formula that is an analog of the so-called Möbius' inversion formula. A heuristic statement is suggested. The generating function for values of function $Ns\,(a,n)$ and also a new characteristic property of prime numbers are corollaries of the statement.

We define the arithmetic function of two variables, $Ns(a,n)$, as follows: n stands for all positive integers and a stands for all non-negative integers,

$$Ns(a,n) = \varphi(n)\frac{\mu\left(\frac{n}{(a,n)}\right)}{\varphi\left(\frac{n}{(a,n)}\right)},$$

where φ is the Euler function, μ is the Möbius function and (a,n) is the greatest common divisor of integers a and n. Generalized character of the $Ns(a,n)$ function is obvious, since

$$Ns(a,n) = \varphi(n), \text{ if } (a,n) = n; \quad Ns(a,n) = \mu(n), \text{ if } (a,n) = 1.$$

Therefore, one can expect that the function $Ns(a,n)$ possesses a number of properties, which are both similar to those of the Euler function and of the Möbius function as well.

Theorem 1.

$$\sum_{d|n} Ns(a,d) = \begin{cases} n\,, & \text{if } n \mid a\ , \\ 0\,, & \text{otherwise,} \end{cases}$$

where the sum is extended to the all divisors d of integer n.

Proof. Let the integer n be represented as $n = n_1 \cdot (a,n)$. Then,

$$\sum_{d|n} Ns(a,d) = \sum_{d=d_1\cdot d_2;\ d_2=(a,d);\ d|n} Ns(a, d_1\cdot d_2) = \sum_{d_1|n_1;\ d_2|(a,n);\ d_2=(a,d_1\cdot d_2)} \varphi(d_1\cdot d_2)\frac{\mu(d_1)}{\varphi(d_1)} =$$

$$= \sum_{d_1|n_1}\mu(d_1) \sum_{d_2=d_2'\cdot d_2'';\ d_2'=(d_1,d_2)=(d_1,(a,n));\ d_2''|\frac{(a,n)}{d_2'}} \frac{\varphi(d_1\cdot d_2'\cdot d_2'')}{\varphi(d_1)} = \sum_{d_1|n_1}\mu(d_1) \sum_{d_2''|\frac{(a,n)}{d_2'};\ d_2'=(d_1,(a,n))} d_2'\varphi(d_2'') =$$

$$= \sum_{d_1|n_1} \mu(d_1)(a,n) = \begin{cases} (a,n) = n, & \text{if } n_1 = 1, \text{ i.e. } n \mid a; \\ 0, & \text{if } n_1 > 1, \text{ i.e. } n \dagger a. \end{cases}$$

Incidently, it follows from the above Proof that

$$\sum_{d|n} \mid Ns(a,d) \mid = (a,n) \cdot 2^k \,,$$

where k is the number of prime divisors of integer $\frac{n}{(a,n)}$.

The statements similar to the so-called Möbius' inversion formulae hold for the function $Ns(a,n)$, [1].

Theorem 2. (An analog of the first Möbius' inversion formula).
Let f is the arithmetic function and

$$g(n) = \sum_{d|n} f(d).$$

Then,

$$\sum_{d|(a,n)} f\left(\frac{n}{d}\right)\, d = \sum_{d|n} Ns(a,d)\, g\left(\frac{n}{d}\right).$$

Proof. We have

$$\sum_{d|n} Ns(a,d) g\left(\frac{n}{d}\right) = \sum_{d|n} Ns(a,d) \sum_{d'|\frac{n}{d}} f(d') = \sum_{d'|n} f(d') \sum_{d|\frac{n}{d'}} Ns(a,d)$$

and, therefore, according to Theorem 1

$$\sum_{d|n} Ns(a,d)\, g\left(\frac{n}{d}\right) = \sum_{\frac{n}{d'}|a} f(d')\frac{n}{d'} = \sum_{d|(a,n)} f\left(\frac{n}{d}\right) d.$$

Corollary 1. ($(a,n) = 1$. It is the first Möbius' inversion formula).

$$f\,(n) = \sum_{d|n} \mu(d)\, g\left(\frac{n}{d}\right).$$

Corollary 2. ($(a,n) = n$; $f(d) = d$).

$$n\ d\,(n) = \sum_{d|n} \varphi\,(d)\, S\left(\frac{n}{d}\right),$$

where $d\,(n)$ is the number of positive divisors of integer n and S is the sum of the positive divisors.

Corollary 3. ($(a,n) = n$; $f\,(d) = 1$).

$$S\,(n) = \sum_{d|n} \varphi\,(d)\, d\left(\frac{n}{d}\right).$$

And so on.

In general, many expressions containing the Möbius function or (and) the Euler function have analogs for the function Ns as well. For instance, the statement (according to Theorem 1)

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\sum_{m=1}^{\infty}\frac{Ns(a,m)}{m^2}=\sum_{k|a}\frac{1}{k};$$

or

$$\sum_{m=1}^{\infty}\frac{Ns(a,m)}{m^2}=\frac{6}{\pi^2}S_{-1}(a),\quad\text{where}\quad S_{-1}(a)\equiv\sum_{d|a}\frac{1}{d},$$

is an analog of the statement [1]

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2}=\sum_{k=1}^{\infty}\frac{c_k}{k^2}=1,$$

where $c_k=\sum_{\ell|k}\mu(\ell)$.

As for summation with respect to a, we note the following property.

Theorem 3. Let $n=p_1^{\alpha_1}...p_k^{\alpha_k}$ is the canonical expansion. Then,

$$\sum_{a=1}^{n}(Ns(a,n))^m=(\varphi(n))^m\prod_{i=1}^{k}\left(1-\frac{1}{(1-p_i)^{m-1}}\right),$$

where m is the non-negative integer.

Proof. It is sufficient to show that

$$\sum_{a=1}^{n}\left(\frac{\mu\left(\frac{n}{(a,n)}\right)}{\varphi\left(\frac{n}{(a,n)}\right)}\right)^m=\left(1-\frac{1}{(1-p)^{m-1}}\right)\sum_{a=1}^{n_1}\left(\frac{\mu\left(\frac{n_1}{(a,n_1)}\right)}{\varphi\left(\frac{n_1}{(a,n_1)}\right)}\right)^m,$$

where $n=n_1p^{\alpha}$; $(n_1,p)=1$ (p is the prime number, $\alpha\geq 1$). We have

$$\sum_{a=1}^{n}\left(\frac{\mu\left(\frac{n}{(a,n)}\right)}{\varphi\left(\frac{n}{(a,n)}\right)}\right)^m=\sum_{1\leq a\leq n;\ (a,p^{\alpha})\geq p^{\alpha-1}}\left(\frac{\mu\left(\frac{n}{(a,n)}\right)}{\varphi\left(\frac{n}{(a,n)}\right)}\right)^m=\sum_{a=1}^{n_1\cdot p}\left(\frac{\mu\left(\frac{n_1\cdot p}{(a,n_1\cdot p)}\right)}{\varphi\left(\frac{n_1\cdot p}{(a,n_1\cdot p)}\right)}\right)^m=$$

$$=\sum_{1\leq a<n_1\cdot p;\ (a,p)=1}\left(\frac{\mu\left(\frac{n_1\cdot p}{(a,n_1\cdot p)}\right)}{\varphi\left(\frac{n_1\cdot p}{(a,n_1\cdot p)}\right)}\right)^m+\sum_{a=1}^{n_1}\left(\frac{\mu\left(\frac{n_1}{(a,n_1)}\right)}{\varphi\left(\frac{n_1}{(a,n_1)}\right)}\right)^m=(p-1)\sum_{a=1}^{n_1}\left(\frac{\mu\left(\frac{n_1\cdot p}{(a,n_1)}\right)}{\varphi\left(\frac{n_1\cdot p}{(a,n_1)}\right)}\right)^m+\sum_{a=1}^{n_1}\left(\frac{\mu\left(\frac{n_1}{(a,n_1)}\right)}{\varphi\left(\frac{n_1}{(a,n_1)}\right)}\right)^m=$$

$$=\left(1-\frac{1}{(1-p)^{m-1}}\right)\sum_{a=1}^{n_1}\left(\frac{\mu\left(\frac{n_1}{(a,n_1)}\right)}{\varphi\left(\frac{n_1}{(a,n_1)}\right)}\right)^m.$$

For negative integers m, obviously, the formula

$$\sum_{a=1}^{n}\left(\frac{\varphi(n)}{\varphi\left(\frac{n}{(a,n)}\right)}\right)^m\left(\mu\left(\frac{n}{(a,n)}\right)\right)^{|m|}=(\varphi(n))^m\prod_{i=1}^{k}\left(1-\frac{1}{(1-p_i)^{m-1}}\right)$$

is valid.

It follows from Theorem 3 that

$$\sum_{a=1}^{n} Ns(a,n) = \begin{cases} 1 & , \quad \text{if } n = 1, \\ 0 & , \quad \text{if } n > 1; \end{cases}$$

and also

$$\sum_{a=1}^{n} \mid Ns(a,n) \mid = \varphi(n)\, 2^k,$$

where k is the number of prime divisors of the integer n.

The following property of the function $Ns(a,n)$ is of particular interest. It is given here in the form of hypothesis because the author has no a completed proof at his disposal.

Theorem 4. Let

$$\prod_{i=1}^{n-1} (1-q^i) = \sum_{k=0}^{\frac{n(n-1)}{2}} p_{n-1}(k)\, q^k.$$

(Coefficients $p_{n-1}(k)$ are of a specified sense in the theory of partitions [2]). Then, for $0 \le a \le (n-1)$

$$Ns(a,n) = \sum_{k \ge 0;\ a+n\cdot k \le \frac{n(n-1)}{2}} p_{n-1}(a+n\cdot k) = \sum_{k=0}^{\left[\frac{n-1}{2} - \frac{a}{n}\right]} p_{n-1}(a+n\cdot k),$$

where $\left[\frac{n-1}{2} - \frac{a}{n}\right]$ is the integral part of the number.

Corollary 1.

$$\varphi(n) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} p_{n-1}(n\cdot k);$$

$$\mu(n) = \sum_{k=0}^{\left[\frac{n-2}{2}\right]} p_{n-1}(1+n\cdot k).$$

Corollary 2. (It expresses a characteristic property of prime numbers.) If p is a prime odd number, then

$$1 + \sum_{k=0}^{\left(\frac{p-1}{2}\right)} p_{n-1}(p\cdot k) = p,$$

and also

$$1 + \sum_{k=0}^{\left(\frac{p-1}{2}-1\right)} p_{n-1}(a+p\cdot k) = 0,$$

where $1 \le a \le (p-1)$.

Corollary 3. (It is a generating function for values of the function $Ns(a,n)$.) Let

$$\frac{(-1)^{n-1}}{1-q^n} \prod_{i=1}^{n-1} (1-q^i) = \sum_{k=0}^{\infty} N_n(k)\, q^k.$$

Then, for $a \ge 1$

$$Ns(a,n) = N_n\left(\frac{(n-1)\,(n-2)}{2} + a - 1\right).$$

References

[1] K.Chandrasekharan, Introduction to analytic number theory, Springer-Verlag, 1968. 140 pp.

[2] George E.Andrews, The Theory of Partitions, Addison-Wesley Publishing Company, 1976. 255 pp.